CS181 Pset 0

Due: Never

The below problems were written so that you can self-assess your comfort with CS 181's mathematical prerequisites. We strongly encourage students who have completed the prerequisites to still complete this optional HW to review concepts that are foundational to the course. The difficulty level of these problems is not intended to reflect the difficulty of future homework assignments.

If you find these problems challenging, please check out the Section 0 document and/or reach out to one of the TFs. Even if you find this HW easy, we still recommend reading through the Section 0 notes for review.

Problem 1

Given the matrix $\mathbf{X}$ and the vectors $\mathbf{y}$ and $\mathbf{z}$ below:

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{1}$$

(a) Expand $\mathbf{X}\mathbf{y} + \mathbf{z}$.

(b) Expand $\mathbf{y}^{\mathbf{T}}\mathbf{X}\mathbf{y}$.

Problem 2

Assume matrix $\mathbf{X}$ has shape $(n \times d)$, and vector $\mathbf{w}$ has shape $(d \times 1)$.

(a) What shape is $\mathbf{y} = \mathbf{X}\mathbf{w}$?

(b) What shape is $(\mathbf{X}^T\mathbf{X})^{-1}$?

(c) Using $\mathbf{y}$ from part (a), what shape is $(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$?

(d) Assume vector $\mathbf{w}' = \mathbf{w}^T$. What shape is $\mathbf{y}' = \mathbf{X}\mathbf{w}'^T$?

Problem 3

Write $\mathbf{u} = \mathbf{u}^{\|} + \mathbf{u}^{\perp}$ where $\mathbf{u}^{\|} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$. is the projection of $\mathbf{u}$ onto $\mathbf{v}$. Prove that $\langle \mathbf{u}^{\|}, \mathbf{u}^{\perp} \rangle = 0$ and that $\mathbf{u} = \mathbf{u}^{\|}$ if and only if $\mathbf{u}$ is a scaled multiple of $\mathbf{v}$.

Problem 4

For an invertible matrix $\mathbf{A}$ show that $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$ where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$.

Problem 5

Solve the following vector/matrix calculus problems. In all of the below, $\mathbf{x}$ and $\mathbf{w}$ are column vectors (i.e. $n \times 1$ vectors). It may be helpful to refer to *The Matrix Cookbook* by Petersen and Pedersen, specifically sections 2.4, 2.6, and 2.7.

(a) Let $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$. Find $\nabla_{\mathbf{x}} f(\mathbf{x}) = \frac{\delta}{\delta \mathbf{x}} f(\mathbf{x})$.

Hint: As a first step, you can expand $\mathbf{x}^T\mathbf{x} = (x_1^2 + x_2^2 + ... + x_n^2)$, where $\mathbf{x} = (x_1, ..., x_n)$.

(b) Let $f(\mathbf{w}, \mathbf{x}) = (1 - \mathbf{w}^T\mathbf{x})^2$. Find $\nabla_{\mathbf{w}} f(\mathbf{w}, \mathbf{x}) = \frac{\delta}{\delta \mathbf{w}} f(\mathbf{w}, \mathbf{x})$.

(c) Let $\mathbf{A}$ be a symmetric n-by-n matrix. If $f(\mathbf{w}, \mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{w}^T\mathbf{x}$, find $\nabla_{\mathbf{x}} f(\mathbf{w}, \mathbf{x}) = \frac{\delta}{\delta \mathbf{x}} f(\mathbf{w}, \mathbf{x})$.

Problem 6

Solve the following:

(a) Verify that $\text{Var}(aX + b) = a^2\text{Var}(X)$.

Hint: As a first step, you can expand $\text{Var}(aX + b)$ using the definition of variance. Simplify using properties of expectations.

(b) Suppose that $X_1, ..., X_n$ are i.i.d., scalar random variables with mean μ and variance σ^2. Let $\bar{X}$ be the mean $\frac{1}{n}\sum_i^n X_i$. Find $\mathbb{E}(\bar{X})$ and $\text{Var}(\bar{X})$.

Problem 7

Prove or come up with counterexamples for the following statements:

(a) Random variables A and B are conditionally independent given C. Does this imply that A and B are unconditionally independent?

(b) Random variables A and B are independent. Does this imply that A and B are conditionally independent given some random variable C?

Problem 8

Suppose you undergo a test for a disease whose frequency in the population is 1% (i.e. the probability of any given person having the disease is 1%). The test for the disease has a 5% false positive rate (i.e. given that you don't have the disease, there's a 5% chance you test positive) and a 10% false negative rate (i.e. given that you do have the disease, there's a 10% chance that you test negative).

Suppose you take the test and it comes back positive. What is the probability that you have the disease?

Problem 9

Show that for scalar random variables X, Y that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$.

Problem 10

Using the probability density function (PDF) of $X \sim \mathcal{N}(0, 1)$ show that X has mean 0 and variance 1.

Hint: The PDF is $p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$. For the mean, you can reason about the properties of the PDF itself to get the answer without integration techniques. For the variance, use integration by parts with $u = x$ and $\text{d}v = xe^{-x^2/2}\text{d}x$ and the fact that the PDF itself integrates to 1.

Problem 11

A random point (X, Y, Z) is chosen uniformly in the ball

$$B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$$

(a) Find the joint PDF of (X, Y, Z).

(b) Find the joint PDF of (X, Y) (this is the marginal distribution on X and Y).

(c) Write an expression for the marginal PDF of X, as an integral.

Problem 12

Suppose we randomly sample a Harvard College student from the undergraduate population. Let X be the indicator of the sampled individual concentrating in computer science, and let Y be the indicator that they work in the tech industry after graduation.

Suppose that the below table represented the joint probability mass function (PMF) of X and Y:

	$Y = 1$	$Y = 0$
$X = 1$	$\frac{10}{100}$	$\frac{5}{100}$
$X = 0$	$\frac{15}{100}$	$\frac{70}{100}$

(a) Calculate the marginal probability $P(Y = 1)$. In the context of this problem, what does this probability represent?

(b) Calculate the conditional probability $P(Y = 1|X = 1)$. In the context of this problem, what does this probability represent?

(c) Are X and Y independent? Why or why not? What is the interpretation of this?

Problem 13

In her most recent work-from-home shopping spree, Nari decided to buy several house plants. She would like for them to grow as tall as possible, but needs your calculus help to understand how to best take care of them.

(a) After perusing the internet, Nari learns that the height y in mm of her Weeping Fig plant can be directly modeled as a function of the oz of water x she gives it each week:

$$y = -3x^2 + 72x + 70$$

Is this function concave, convex, or neither? Explain why or why not.

(b) Solve analytically for the critical points of this expression (i.e., where the derivative of the function is zero). For each critical point, use the second-derivative test to identify if each point is a max or min point, and use arguments about the global structure (e.g., concavity or convexity) of the function to argue whether this is a local or global optimum.

(c) How many oz per week should Nari water her plant to maximize its height? With this much water how tall will her plant grow?

(d) Nari also has a Money Tree plant. The height y in mm of her Money Tree can be directly modeled as a function of the oz of water x she gives it per week:

$$y = -x^4 + 16x^3 - 93x^2 + 230x - 190$$

Is this function concave, convex, or neither? Explain why or why not.

Credits: Problems 11 and 12 were inspired by Exercise 7.19 and Example 7.1.5 in Blitzstein & Hwang's "Introduction to Probability".